ISYE 7201: Production & Service Systems
Spring 2020
Instructor: Spyros Reveliotis
1st Midterm Exam (Take Home)
Release Date: February 3, 2020
Due Date: February 10, 2020

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): Let $X = \langle X_n,\ n \in Z_0^+\rangle$ and $Y = \langle Y_n,\ n \in Z_0^+\rangle$ and $W = \langle W_n,\ n \in Z_0^+\rangle$ be stochastic processes such that $Y_n = X_n^2$ and $W_n = X_n^3$, for all n. If X is a discrete-time Markov chain, determine whether Y and W also preserve this property. For each of these two processes, provide a rigorous proof in case of a positive answer; otherwise, provide a counter-example.

Problem 2 (20 points): Consider the "unit" cube in the positive orthant , i.e., the cube with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0,), (1,0,1), (0,1,1) and (1,1,1). An agent moves on the vertices of this cube according to the following rule: Let X_t be the position of this agent at period, for $t = 0, 1, \ldots$. Then, X_{t+1} is one of the three neighboring vertices to vertex X_t in the considered cube, and each of the three possible transitions are equally likely.

i. **(5 pts)** Provide the state transition diagram for the DTMC that is defined by the agent motion.

ii. **(7.5 pts)** Let Y_t denote the distance of X_t from the origin $(0, 0, 0)$ in terms of the smallest number of edges of the considered unit cube that must be traversed in order to get from $(0, 0, 0)$ to X_t. Argue that Y_t is also a DTMC.

iii. **(7.5 pts)** Use the result of part (ii) in order to compute the mean recurrence time of vertex $(0, 0, 0)$ in the dynamics of X.

Problem 3 (20 points) – A simple stochastic optimization problem Consider a discrete-time stochastic process that when it is in control it generates a profit of 0.75 units per period in a certain currency. However, at every period the process can get out of control with probability 1/2. At every period that the process is out of control, we can try to bring it back in control by expending a monetary value of x for some $x \in [0, 1]$. The corresponding probability of success is $\sqrt{x}$. Your task is to determine the value of x that maximizes the average profit rate for this process over an infinite operational horizon.

Problem 4 (20 points) – A "state space reduction" problem Consider an irreducible and aperiodic DTMC X with state space $S_X = \{1, 2, 3\}$, and the (symbolic) function $f : S_X \to \{a, b\}$ defined as follows: $f(1) = f(2) = a; f(3) = b$. Furthermore, consider the stochastic process Y with $Y_n = f(X_n),\ n = 0, 1, 2, \ldots$ and let $S_Y = \{a, b\}$ denote the state space of Y.

i. **(5 pts)** Argue that the CTMC X has a limiting distribution $\pi = (\pi_1, \pi_2, \pi_3)$.

ii. **(5 pts)** Show that the limiting distribution π of X implies a limiting distribution ϕ for Y.

iii. **(5 pts)** Construct a DTMC Z defined on the state space S_Y that has the same limiting distribution ϕ with process Y.

iv. **(5 pts)** Let r_X be a "reward" function that is defined on the state space S_X; i.e., process X collects the reward $r_X(X_n)$ at period $n = 0, 1, 2, \ldots$. Define a "reward" function r_Y on S_Y such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} r_X(X_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} r_Y(Y_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} r_Y(Z_n)$$

.

Problem 5 (20 points): In the paper

> J. J. Bartholdi and D. D. Eisenstein, *"A production line that balances itself"*, OR, vol. 44, no. 1, 1996

that introduced the original analysis of the "Bucket Brigades" policy, the authors remark that (a) the policy dynamics are of a "pull" nature where the behavior of the last (and fastest) picker in the line drives the behavior of every other picker, and (b) it is also this effect that, at the end, determines the convergence of the entire policy that is established in the paper.

Your task in this problem is to provide an analytical substantiation to this claim. You can do this by taking the following steps:

i. **(5 pts)** Use the recursion for the vector $\mathbf{a}^{(t)}$ that was developed in class in order to express every component $\mathbf{a}_i^{(t+1)}$, $i = 1, \ldots, n$, of the vector $\mathbf{a}^{(t+1)}$ as a function of the subsequence its last component $\langle \mathbf{a}_n^{(k)},\ k = 0, 1, \ldots, t\rangle$, that concerns the last picker.

ii. **(10 pts)** Use the results of part (i) in order to develop an alternative argument for the convergence of the sequence $\langle \mathbf{a}_n^{(k)},\ k = 0, 1, \ldots\rangle$.

iii. **(5 pts)** Finally, argue that the convergence of $\langle \mathbf{a}_n^{(k)},\ k = 0, 1, \ldots \rangle$ implies the convergence of $\langle \mathbf{a}_i^{(k)},\ k = 0, 1, \ldots \rangle$, for every other $i \in \{1, \ldots, n-1\}$.